CS Inequality Overview

The Cauchy–Schwarz inequality (also called Cauchy–Bunyakovsky–Schwarz inequality)

Statement:

Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in \mathbb{C}$ then we have the following:

$$\left|\sum_{i=1}^{n} a_i \bar{b}_i\right|^2 \leq \left(\sum_{i=1}^{n} |a_i|^2\right)\left(\sum_{i=1}^{n} |b_i|^2\right)$$

and equality holds iff $a_i = tb_i$ for some $t \in \mathbb{C}\ \forall i \in \{1, 2, 3, \cdots n\}$

Proof:

Consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by

$$f(t) = \sum_{i=1}^{n} |a_i - tb_i|^2 \quad (Remember) \tag{1}$$

Note:

$$|z|^2 = z\overline{z} \tag{2}$$

Using (2) in (1) we get,

$$\begin{aligned} f(t) &= \sum_{i=1}^{n} (a_i - tb_i)(\overline{a_i - tb_i}) \\ &= \sum_{i=1}^{n} (a_i - tb_i)(\overline{a_i} - \overline{t}\overline{b_i}) \end{aligned}$$

$$= \sum_{i=1}^{n} \left(|a_i|^2 - (a_i\overline{b_i}\overline{t} + \overline{a_i}b_i t) + |b_i||t|^2\right)$$

Now, putting $t = x + iy$ in the above we get the following:

$$f(x+iy) = \left(\sum_{i=1}^{n} |a_i|^2\right) - \left(\sum_{i=1}^{n} (a_i \overline{b_i}(\overline{x+iy}) + \overline{a_i} b_i (x+iy))\right) + \left(\sum_{i=1}^{n} |b_i|\right) |t|^2$$

$$= A - 2x\Re(B) - i2y\Im(B) + C(x^2+y^2) \quad (\text{where } A = \left(\sum_{i=1}^{n} |a_i|^2\right)$$

$$B = \sum_{i=1}^{n} a_i \bar{b}_i \ \&\ C = \left(\sum_{i=1}^{n} |b_i|\right))$$

Now, completing the squares and organising terms we get,

$$C\left(\left(x - \frac{\Re(B)}{C}\right)^2 + \left(y - \frac{\Im(B)}{C}\right)^2\right) + A - \frac{|B|^2}{C} = f(t) \tag{3}$$

From equation (1) we get $f(t) \geq 0 \ \forall t \in \mathbb{C}$ and hence from equation (3) we get that for $t = \frac{\Re(B)}{C} + i\frac{\Im(B)}{C}$ also $f(t) \geq 0$. That is $A - \frac{|B|^2}{C} \geq 0 \Rightarrow |B|^2 \leq AC$.